

Properties of Logarithmic Functions

Product Rule: $\log_a (MN) = \log_a M + \log_a N$

Quotient Rule: $\log_a \left(\frac{M}{N} \right) = \log_a M - \log_a N$

Power Rule: $\log_a (M)^r = r \log_a M$

***Note:** If $M = N$ then $\log_a M = \log_a N$, if $\log_a M = \log_a N$, then $M = N$,

and if $M^x = M^y$, then $x = y$.

See pg. 169 for examples.

Example: Proving the Product Rule for Logarithms

Prove $\log_b (RS) = \log_b R + \log_b S$.

Solution: Let $x = \log_b R$ and $y = \log_b S$. The corresponding statements are

$$b^x = R \text{ and } b^y = S. \text{ So,}$$

$$RS = b^x \cdot b^y$$

$$= b^{x+y} \quad \text{first prop. of exponents}$$

$$\log_b (RS) = x + y \quad \text{change to logarithmic function}$$

$$= \log_b R + \log_b S \quad \text{def. of } x \text{ and } y$$

Expanding the Logarithm of a Product

Example:

Assume x and y are positive, use properties of logarithms to write $\log(27x^2y^5)$ as a sum of logarithms or multiples of logarithms.

$$\begin{aligned}
 \log(27x^2y^5) &= \log 27 + \log x^2 + \log y^5 && \text{product rule} \\
 &= \log 3^3 + \log x^2 + \log y^5 && 27 = 3^3 \\
 &= 3 \log 3 + 2 \log x + 5 \log y && \text{power rule}
 \end{aligned}$$

Expanding the Logarithm of a Quotient

Example:

Assuming x is positive, use properties of logarithms to write $\ln \left(\frac{\sqrt{x^4 + 2}}{x} \right)$ as a sum or difference of logarithms or multiples of logarithms.

$$\begin{aligned}
 \ln \left(\frac{\sqrt{x^4 + 2}}{x} \right) &= \ln \frac{(x^4 + 2)^{\frac{1}{2}}}{x} \\
 &= \ln (x^4 + 2)^{\frac{1}{2}} - \ln x && \text{quotient rule} \\
 &= \frac{1}{2} \ln (x^4 + 2) - \ln x && \text{power rule}
 \end{aligned}$$

Condensing a Logarithmic Expression

Example:

Assuming x and y are positive, write $\ln x^2 - 3 \ln (xy)$ as a single logarithm.

$$\begin{aligned}
 \ln x^2 - 3 \ln (xy) &= \ln x^2 - \ln (xy)^3 && \text{power rule} \\
 &= \ln x^2 - \ln (x^3 y^3) \\
 &= \ln \frac{x^2}{x^3 y^3} && \text{quotient rule} \\
 &= \ln \frac{1}{xy^3}
 \end{aligned}$$

Change of Base Formula for Logarithms

For positive real numbers a , b and x with $a \neq 1$ and $b \neq 1$,

$$\log_b x = \frac{\log_a x}{\log_a b} \quad \text{OR} \quad \log_a x = \frac{\log x}{\log a} \quad \text{OR} \quad \log_a x = \frac{\ln x}{\ln a}$$

Evaluating Logarithms by Changing the Base

Example:

$$a) \log_3 18 = \frac{\ln 18}{\ln 3} = 2.6309... \approx 2.63$$

$$b) \log_4 20 = \frac{\log 20}{\log 4} = 2.1609... \approx 2.16$$

Logarithmic Functions with $b > 1$, $f(x) = \log_b x$

Domain $(0, \infty)$

Range: All reals

x-intercept = $(1, 0)$

y-intercepts: none

Increasing on its domain

No symmetry: neither even nor odd

No local extrema

No horizontal asymptotes

Vertical asymptote: $x = 0$

End behavior: $\lim_{x \rightarrow \infty} \log_b x = \infty$

Passes through $(1, 0)$ and $(b, 1)$

One-to-one

